# The Arithmetic and Geometry of Elliptic Surfaces

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**Abstract:** We survey some aspects of the theory of elliptic surfaces and give some results aimed at determining the Picard number of such a surface. For the surfaces considered, this will be equivalent to determining the Mordell-Weil rank of an elliptic curve defined over a function field in one variable. An interesting conjecture concerning Galois actions on the relative de Rham cohomology of these surfaces is discussed.

This paper focuses on an important class of algebraic surfaces called elliptic surfaces. The results while geometric in character are arithmetic at heart, and for that reason we devote a fair portion of our discussion to those definitions and facts that make the arithmetic clear. Later in the paper, we will explain some recent results and conjectures. This is a preliminary version, the detailed version will appear elsewhere.

There are a number of natural routes leading to the definition of the class of elliptic surfaces. Let $E$ denote a compact connected complex manifold with $\dim_{\mathbb{C}} E = 2$.

**Theorem 1:** (Siegel) The field of meromorphic functions on $E$ has transcendence degree $\leq 2$ over $\mathbb{C}$, i.e. the field of meromorphic functions is:

1) $\mathbb{C}$ constant functions

2) a finite separable extension of $\mathbb{C}(x)$

3) a finite separable extension of $\mathbb{C}(x, y)$. $\qquad \blacksquare$

Case 3) is precisely the set of algebraic surfaces, i.e. those admitting an embedding into $\mathbb{P}_{\mathbb{C}}^N$. Case 2) was studied by Kodaira, leading to a series of three papers:

Kodaira, K., "On complex analytic surfaces I, II, III," Annals of Math. 77 and 78, 1963, which expound on elliptic surfaces. Kodaira makes the following definition:

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**Definition 2:** $E$ is *elliptic* if:

1) $\exists$ a smooth curve (read Riemann surface) $X$ and a proper holomorphic map $\pi\colon E \to X$ of $E$ onto $X$ such that

2) $\pi^{-1}(x)$ (with multiplicity) is a non-singular curve $E_x$ of genus one, i.e. a torus, for general $x \in X$. ("General" means for all but finitely many $x \in X$.)

**Theorem 3:** (Kodaira) Transcendence degree $= 1$ (case 2) above) implies $E$ is elliptic. $\quad\square$

There are of course many elliptic surfaces which are algebraic and so have transcendence degree $= 2$.

From now on $E$ will denote an elliptic surface. One immediate question is to determine the nature of the singular fibers of $\pi\colon E \to X$ at the finite set of points $S = \{x_1, \ldots, x_n\} \subset X$ where the fiber $\pi^{-1}(x_i) = E_{x_i}$ is something other than a non-singular curve (occuring with multiplicity one) of genus one. In the second of the above mentioned papers of Kodaira, a complete description of the singular fiber types is given. (We will ignore multiple fibers, as these can't occur for the type of elliptic surface defined below.)

For our purposes, we will narrow the definition of elliptic surface as follows:

**Definition 4:** A compact connected complex surface $E$ will be called *elliptic* if:

1) $\exists$ a smooth curve $X$ (read Riemann surface) and a proper holomorphic map $\pi\colon E \to X$ mapping $E$ onto $X$ such that

2) $\pi^{-1}(x)$ is a non-singular curve of genus one for general $x \in X$, and

3) $\pi\colon E \to X$ has a section, i.e. $\exists$ a holomorphic map $\mathcal{O}\colon X \to E$ s.t. $\pi \circ \mathcal{O} = 1_X$,

4) $E/X$ is relatively minimal, i.e. there are no exceptional curves of the first kind in the fibers,

5) $E/X$ is not isotrivial.

A few comments are in order. First condition 3) forces $E$ to be algebraic and devoid of multiple singular fibers. The section $\mathcal{O}\colon X \to E$ furnishes a $K(X)$-rational point on the generic fiber $E^{\text{gen}}$ viewed as a curve over $K(X)$. Thus the generic fiber $E^{\text{gen}}$ is an elliptic curve over the field $K(X)$ of meromorphic/rational functions on $X$. Assumption 5) means

that we have a non-trivial variation of complex structure in the "good" fibers. Simply put, this means that the $J$-invariant of the fibers, which can be viewed as a meromorphic/rational function on $X$, is non-constant. We denote this function by $\mathcal{J} \in K(X)$. The fact that $\mathcal{J}$ is non-constant allows us, via the Mordell-Weil theorem, to conclude that the group $E^{\mathrm{gen}}(K(X))$ of $K(X)$-rational points on the generic fiber, or what is the same, the group of sections of $\pi\colon E \to X$, is a finitely generated abelian group. We denote its rank by $r_E$. Finally, condition 4) in the definition implies that we have blown down all the exceptional curves in the fibers of $\pi\colon E \to X$. $E$ is then the unique minimal compactification of the so-called Néron model of the elliptic curve $E^{\mathrm{gen}}/K(X)$. We remark that the map $\pi$ and the curve $X$ are essentially uniquely determined by the fact that the Jacobian of $X$ must be the Albanese of $E$.

This places us in a situation analogous to the common arithmetic situation where $E$ is an elliptic curve over a number field $K$, where the Néron model is an arithmetic surface over the "curve" $\mathrm{Spec}(\mathcal{O}_K)$, $\mathcal{O}_K$ being the ring of algebraic integers in $K$, and where the fiber over a point in $\mathrm{Spec}\,\mathcal{O}_K$ is the reduction of $E$ modulo a nonzero prime ideal $\varphi \subset \mathcal{O}_K$. The finitely generated abelian group $E(K)$ describes the solutions in $K$ to the Diophantine equation(s) defining $E$.

Classical Examples:

|  |  |  |
|---|---|---|
| Legendre (Level 2) | $Y^2 = X(X-1)(X-\lambda)$ | avoid characteristic 2<br>singular fibers at $\lambda = 0, 1, \infty$ |
| Level 3 | $X^3 + Y^3 + 1 = \mu XY$ | avoid characteristic 3<br>singular fibers at $\mu^3 = 27$ or $\infty$ |

Further Examples (Elliptic Modular Surfaces):

In a paper entitled "On elliptic modular surfaces", which appears in the Journal Math. Soc. Japan, Vol. 24, No. 1 (1972), T. Shioda constructs an important class of elliptic surfaces. Given a subgroup of finite index $\Gamma \subset SL_2(\mathbf{Z})$ with $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \notin \Gamma$, Shioda constructs a family of elliptic curves $E_\Gamma$ over $X_\Gamma$ the modular curve $\Gamma \backslash \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ with the obvious monodromy representation given by $\Gamma$ and where the lattice of periods of the fiber over $x \in X_\Gamma$ is homothetic to $\mathbf{Z}\tau + \mathbf{Z}$ where $\tau \in \mathbb{H}$, the complex upper half-plane, corresponds to $x \in X_\Gamma$.

This leads us naturally into the world of classical automorphic forms. We will allude to this in several other places. For now, we content ourselves with recalling one of Shioda's results, namely that the space of cusp forms of weight three for $\Gamma$, $S_3(\Gamma)$, is naturally isomorphic to the space of holomorphic two forms on $E$, $H^0(E, \Omega_E^2)$.

We now turn to the main object of interest in this paper:

**Definition 5:** The Néron-Severi group $NS(E)$ of $E$ is defined to be the group of divisors of $E$ modulo algebraic equivalence (as opposed to rational or linear equivalence):

$$NS(E) = \frac{\text{divisors}}{\text{alg. equiv. to } 0} \subset H^2(E, \mathbf{Z}).$$

We remark that for these surfaces algebraic is the same as homological equivalence, so the Néron-Severi group sits inside $H^2(E, \mathbf{Z})$, which can be shown to be torsion-free. The Picard number is defined to be

$$\rho_E = \text{rank } NS(E).$$

By the Lefschetz Theorem on (1,1)-classes one also has:

$$NS(E) = H^{1,1} \cap H^2(E, \mathbf{Z}) \subset H^2(E, \mathbb{C})$$

or

$$NS(E) = \text{the group of topological } \mathbb{C}\text{-line bundles}$$

$$\text{which admit analytic structure}$$

PROBLEM: Calculate $\rho_E$.

Because we are interested only in the rank of the Néron-Severi group, it is reasonable to tensor with $\mathbb{Q}$ and work with

$$NS(E) \otimes_{\mathbf{Z}} \mathbb{Q}.$$

Now away from the "bad" fibers over $S = \{x_1 \ldots x_n\} \subset X$ the family $E|_{X-S} = \pi^{-1}(X-S)$ is locally differentiably trivial, so it is natural to use the Leray spectral sequence

$$E_2^{p,q} = H^p(X, R^q \pi_* \mathbb{Q}) \Rightarrow H^{p+q}(E, \mathbb{Q})$$

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to understand $H^2(E,\mathbb{Q})$ in terms of the base $X$ and the fibers which are tori. The Leray spectral sequence degenerates at $E_2$ and yields a filtration

$$0 \subset F_{\mathbb{Q}}^2 \subset F_{\mathbb{Q}}^1 \subset F_{\mathbb{Q}}^0 = H^2(E,\mathbb{Q})$$

where

$$F_{\mathbb{Q}}^1 = \ker(H^2(E,\mathbb{Q}) \to H^0(X, R^2\pi_*\mathbb{Q}))$$

consists of classes which restrict to zero on each fiber, and

$$F_{\mathbb{Q}}^2 = \mathrm{im}(H^2(X,\mathbb{Q}) \xrightarrow{\pi^*} H^2(E,\mathbb{Q})) = \mathbb{Q}[E_{x_0}]$$

is generated by the cohomology class of a fiber.

The filtration quotient is

$$F_{\mathbb{Q}}^1/F_{\mathbb{Q}}^2 \cong H^1(X, R^1\pi_*\mathbb{Q}).$$

Now the Hodge decomposition on $H^*(E,\mathbb{C})$ induces a Hodge structure on the filtration quotient above

$$H^1(X, R^1\pi_*\mathbb{Q}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$$

where $H^{2,0}$ is all of $H^0(E, \Omega_E^2)$ (because the restriction of a holomorphic two form to a fiber is necessarily zero when the fiber is a curve).

Now there is a well-known

**Theorem 6:** (Shioda) $\rho_E = r_E + 2 + \sum_{s \in S}(m_s - 1)$ where $m_s$ is the number of irreducible components making up the fiber $E_s = \pi^{-1}(s)$. Thus the geometric quantity $\rho_E$ is essentially the arithmetic quantity $r_E$, the rank of the Mordell-Weil group of the generic fiber $E^{\mathrm{gen}}$ treated as an elliptic curve over the field $K(X)$. $\quad\square$

In practice the numbers $m_s$ are easy to determine, it is $r_E$ that is in general impossible to compute. What should we focus on?

Let $V_{\mathbb{Q}}^i$ be the span of the algebraic cycles in $F_{\mathbb{Q}}^i$ so

$$V_{\mathbb{Q}}^i = (NS(E) \otimes_{\mathbb{Z}} \mathbb{Q}) \cap F_{\mathbb{Q}}^i$$

and let $W_{\mathbb{Q}} = V_{\mathbb{Q}}^1 / V_{\mathbb{Q}}^2 = \dfrac{(NS(E) \otimes_{\mathbb{Z}} \mathbb{Q}) \cap F_{\mathbb{Q}}^1}{F_{\mathbb{Q}}^2} \subset H^1(X, R^1\pi_*\mathbb{Q})$.

**Theorem 7:** $\dim_{\mathbb{Q}} W_{\mathbb{Q}} = r_E$. $\square$

**Proof:** See Stiller [5].

It therefore behooves us to look at $R^1\pi_*\mathbb{Q}$ or $R^1\pi_*\mathbb{C}$ which over $X - S$ is a locally constant sheaf of rank two, so that

$$R^1\pi_*\mathbb{C}|_{X-S} \otimes_{\mathbb{C}} \mathcal{O}_{X-S}$$

is a rank two holomorphic vector bundle on $X - S$. Let's study this bundle.

To get quickly to the heart of the matter we adopt a naive point of view.

Pick a base point $x_0 \in X - S$. Here $S$ will contain the support of the "bad" fibers and some additional points to be named later.

In a sufficiently small neighborhood $U$ of $x_0$

$$\pi^{-1}(U) = E|_U$$

$$\downarrow \pi|_{\pi^{-1}(U)}$$

$$U$$

$E|_U$ is a $\mathcal{C}^\infty$-trivial fiber bundle, i.e.

$$
\begin{array}{ccc}
\pi^{-1}(U) & \cong & U \times \pi^{-1}(x_0) \\
\downarrow \pi|_{\pi^{-1}(U)} & & \downarrow pr_1 \\
U & = & U
\end{array}
.
$$

Write $E_{x_0}$ for $\pi^{-1}(x_0)$ and choose an oriented basis $\gamma_1, \gamma_2 \in H_1(E_{x_0}, \mathbb{Z}) \cong \mathbb{Z}^2$ for the homology of the fiber and consider

$$\omega_i(x) = \int_{\gamma_i} \Omega|_{E_x}$$

where $\Omega$ is an appropriate meromorphic 1-form on $E$ with poles only on the vertical fibers, i.e. $\Omega|_{E^{\mathrm{gen}}}$ is a $K(X)$-rational differential of the $1^{\mathrm{st}}$ kind on the curve $E^{\mathrm{gen}}/K(X)$. Notice that for an appropriate finite set of points $S$ which includes the support of the singular

fibers, the functions $\omega_i(x)$ can be analytically continued as holomorphic non-vanishing functions throughout $X - S$. Moreover,

$$\text{Im } \omega_1(x)/\omega_2(x) > 0.$$

For a path $\gamma \in \pi_1(X - S, x_0)$, analytic continuation of the pair $\omega_1, \omega_2$ around $\gamma$ yields

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \longmapsto M_\gamma \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

where $M_\gamma \in SL_2(\mathbf{Z})$. This is called the *monodromy representation* of $E/X$ and the image of the fundamental group in $SL_2(\mathbf{Z})$ is a subgroup $\Gamma$ of finite index in $SL_2(\mathbf{Z})$ which is unique up to conjugation in $SL_2(\mathbf{Z})$. This group does not depend on the choice of $\Omega$ or on $S$ (provided $S$ contains the support of the singular fibers). $\Gamma$ is called the *monodromy group* of $SL_2(\mathbf{Z})$.

Recall (see Deligne SLN 163) that the following notions are equivalent:

1) a representation of $\pi_1(X - S, x_0) \longrightarrow GL_2(\mathbb{C})$

2) a local system (locally constant sheaf) $V$ of rank 2 on $X - S$

3) a rank 2 holomorphic vector bundle $\mathcal{E}_0 = V \otimes_\mathbb{C} \mathcal{O}_{X-S}$ over $X - S$ with integrable holomorphic connection $D_0$

$$\mathcal{E}_0 \xrightarrow{D_0} \mathcal{E}_0 \otimes_{\mathcal{O}_{X-S}} \Omega^1_{X-S}$$

having regular singular points.

We recall that $\mathcal{E}_0$ can be uniquely extended to a holomorphic (algebraic) rank 2 bundle $\mathcal{E}$ on $X$ together with a meromorphic (rational) connection $D$ having regular singular points. This is known as the *Gauss-Manin connection*.

4) A second order linear differential operator $\Lambda$ rational$/K(X)$ with regular singular points. In our case $\Lambda \omega_i = 0$   for $i = 1, 2$, i.e. $\Lambda$ annihilates the periods of $\Omega$ as functions on the base. This is the *Picard-Fuchs equation* of $E/X$ and $\omega_1, \omega_2$ form a basis for the two dimensional space of solutions at $x_0$, and elsewhere via analytic continuation.

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Our point of view now shifts to this differential equation ($R^1\pi_*\mathbb{C}|_{X-S}$ is the $V$ above). We ask "How much information can we recover from $\Lambda$?".

**Theorem 8:** $E$ is determined by $\Lambda$ up to generic isogeny. We remark that $\Lambda$ depends on the choice of $\Omega$ but any other choice is $g\Omega$ for $g \in K(X)$ and this transforms $\Lambda$ in the obvious simple way. (See Stiller [2]).    $\square$

**Theorem 9:** If $E/X$ and $E'/X$ are generically isogeneous elliptic surfaces over a fixed base curve $X$ then

1) $[PSL_2(\mathbb{Z}){:}\overline{\Gamma}] = [PSL_2(\mathbb{Z}){:}\ \overline{\Gamma}']$

2) $b_i(E) = b_i(E')$   $i = 0, \ldots, 4$   Betti numbers

3) $p_g(E) = p_g(E')$ and $q(E) = q(E') = $ genus $X$

4) $\rho_E = \rho_{E'}$

5) $r_E = r_{E'}$

Remark on the proof: 5) is immediate and is used to prove 4) via the formula

$$\rho_E = r_E + 2 + \sum_{s \in S}(m_s - 1).$$

What is interesting is that $m_s$ is not preserved by generic isogeny – only the sum is. In particular a generic isogeny

$$
\begin{array}{ccc}
E & \xrightarrow{\ \phi\ } & E' \\
{\scriptstyle \pi}\searrow & & \swarrow{\scriptstyle \pi'} \\
& X &
\end{array}
$$

as a rational map which is an isogeny of the fibers almost everywhere, may not extend to all of $E$ as a regular map, and the same may hold for the dual isogeny

$$
\begin{array}{ccc}
E' & \xrightarrow{\ \phi'\ } & E \\
{\scriptstyle \pi'}\searrow & & \swarrow{\scriptstyle \pi} \\
& X &
\end{array}
\quad .
$$

The sum of the $m_s$ can be determined using the exponents at $s$ of $\Lambda$ (which is easily obtainable local information) independent of the isogeny class. For details see Stiller [3].    $\square$

Now how do we capture $H^1(X, R^1\pi_*\mathbb{C})$ in terms of $\Lambda$? We take our clue from Manin. Given any section $s\colon X \to E$, $\pi \circ s = 1_X$, we can locally (say near $x_0 \in X - S$) take a family of paths $\gamma_x$ between the points $s(x)$ and $\mathcal{O}(x)$ on the fiber $E_x$ and compute

$$f(x) = \int_{\gamma_x} \Omega|_{E_x}$$

which is defined up to the periods

$$f + m\omega_1 + n\omega_2.$$

Since $\Lambda$ annihilates the periods,

$$\Lambda f = Z$$

turns out to be a well-defined rational function. $f$ is thus annihilated by a 3rd order operator $\tilde{\Lambda}$ and the rank 3 local system $V_{\tilde{\Lambda}}$ arises an element of $\mathrm{Ext}^1(\mathbb{C}, V_\Lambda)$ where $\mathbb{C}$ is the trivial local system. The monodromy representation for $\tilde{\Lambda}$ takes the form

$$\gamma \in \pi_1(X - S, x_0) \longmapsto \begin{pmatrix} 1 & m_\gamma \ n_\gamma \\ 0 & M_\gamma \end{pmatrix} \in SL_3(\mathbb{Z}).$$

The motivation here is that the cohomology class of the algebraic cycle $s - \mathcal{O}$ is essentially zero on the fibers, and so provides an element in the Leray filtration quotient

$$F^1_{\mathbb{C}}/F^2_{\mathbb{C}} = H^1(X, R^1\pi_*\mathbb{C}).$$

We make the following definitions:

**Definition 10:** $Z \in K(X)$ is *exact* if $\Lambda f = Z$ has a global single-valued meromorphic solution. Thus $Z \in \Lambda K(X) \subset K(X)$.

**Definition 11:** $Z \in K(X)$ is *locally exact* if $\forall p \in X$ the equation $\Lambda f = Z$ restricted to a small neighborhood $U_p$ of $p$ has a single-valued meromorphic solution. We denote the set of locally exact $Z \in K(X)$ by $L_\Lambda^{\mathrm{para}}$. (The notation comes from the theory of automorphic forms and the notion of parabolic cohomology).

**Definition 12:** $\mathrm{H}^1_{IDR}$ is defined to be $L_\Lambda^{\mathrm{para}}/\Lambda K(X)$ and is called the *inhomogeneous de Rham cohomology*.

Some remarks are in order. First we have slid over the non-intrinsic nature of $\Lambda$ which depends on the choice of $\Omega$ and on the choice of a derivation $\frac{d}{dx}$ on $K(X)$. A more intrinsic formulation would treat $Z$ as $Z(dx)^2$ a meromorphic quadratic differential, i.e. a meromorphic section of $(\Omega_X^1)^{\otimes 2}$. In any event $H_{IDR}^1$ is independent of any choices. Secondly, local exactness can be formulated as a residue condition.

**Theorem 13:** $H_{IDR}^1$ is canonically isomorphic to $H^1(X, R^1\pi_*\mathbb{C})$.

The proof is achieved by showing that both groups are naturally isomorphic to the subgroup of locally split extensions in $\mathrm{Ext}^1(\mathbb{C}, V_\Lambda)$. Note that

$$\mathrm{Ext}^1(\mathbb{C}, V_\Lambda) \cong H^1(\pi_1(X - S, x_0), (V_\Lambda)_{x_0}) \cong H^1(X - S, R^1\pi_*\mathbb{C}|_{X-S})$$

with the middle group being the usual group cohomology. The locally split classes correspond to parabolic cohomology and $H^1(X, R^1\pi_*\mathbb{C})$. Here $H^1(X, R^1\pi_*\mathbb{C}) \hookrightarrow H^1(X_0, R^1\pi_*\mathbb{C}|_{X_0})$ where $X_0 = X - S$. This last inclusion comes from the exact sequence of low order terms in the Leray spectral sequence for $i\colon X_0 \hookrightarrow X$ and the sheaf $R^1\pi_*\mathbb{C}|_{X_0}$. (See Stiller [4], [5]). ☐

Now that we have identified $H_{IDR}^1$ with $H^1(X, R^1\pi_*\mathbb{C})$ what about the Hodge decomposition on the latter.

**Theorem 14:** There are two divisors $\mathcal{A}_0 < \mathcal{A}$ on $X$, easily computable in terms of the local behavior of $\Lambda$, such that every element of $L(\mathcal{A}_0)$ is locally exact but never exact (unless 0) and such that no locally exact element in $L(\mathcal{A}) \cap L_\Lambda^{\mathrm{para}}$ is ever exact (except 0), and such that

$$L(\mathcal{A}_0) \hookrightarrow H_{IDR}^1$$

corresponds to $H^{2,0}$ in $H^1(X, R^1\pi_*\mathbb{C})$ and

$$L(\mathcal{A}) \cap L_\Lambda^{\mathrm{para}} \hookrightarrow H_{IDR}^1$$

corresponds to $H^{2,0} \oplus H^{1,1}$. ☐

The point of this result is that we now have <u>unique representatives</u> of the form $\Lambda f = Z$ for elements of $H^{2,0}$ and $H^{2,0} \oplus H^{1,1}$ in $H^1(X, R^1\pi_*\mathbb{C})$. (See Stiller [5]).

**Application:**

We assume for the moment that $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \notin \Gamma \subset SL_2(\mathbf{Z})$ where $\Gamma$ is the monodromy group of $E/X$. We then have a diagram

$$
\begin{array}{ccc}
E \xrightarrow{\sim} E_\Gamma \times_{X_\Gamma} X & \longrightarrow & E_\Gamma \\
\pi \searrow \quad \downarrow & & \downarrow \pi_\Gamma \\
X & \xrightarrow{\omega} & X_\Gamma
\end{array}
$$

where $\omega = \omega_1/\omega_2$ is the so-called *period map*. If we suppose $X$ is Galois over the modular curve $X_\Gamma$ then $G = \mathrm{Gal}(X/X_\Gamma)$ acts on $H^1(X, R^1\pi_*\mathbb{Q})$ and preserves Hodge type in $H^1(X, R^1\pi_*\mathbb{C})$.

**Problem:** Let $V$ be an irreducible rational or complex representation of $G$. What is the multiplicity of $V$ in $H^1(X, R^1\pi_*\mathbb{Q})$ or $H^1(X, R^1\pi_*\mathbb{C})$?

We have been able to show in many cases where $G$ is cyclic, i.e. $K(X)$ is a cyclic extension of the field of modular functions $K(X_\Gamma)$, that all multiplicities are one. This can't be true in general, but we conjecture that it holds when $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \notin \Gamma$ under suitable hypotheses on $G$ (modulo the obvious trivial constituents from $H^0(X_\Gamma, \Omega^2_{X_\Gamma})$ etc.). When the multiplicities are all 1, we can explicitly decompose the $G$-modules $H^{2,0}$ and $H^{2,0} \oplus H^{1,1}$ using our unique representatives. Since Hodge type is preserved and the multiplicities are one, $H^{1,1}$ is the sum of those irreducible constituents of $H^{2,0} \oplus H^{1,1}$ not in $H^{2,0}$. If in turn all the complex irreducible constituents (say $G$ is abelian – so that over $\mathbb{C}$ all irreducible constituents $V$ are one dimensional, and over $\mathbb{Q}$ we want eigenvalues which are all primitive $d^{\text{th}}$ roots of one) of a given irreducible rational representation (dimension $\phi(d)$ in the abelian case) lie in the $H^{1,1}$ part, we get a contribution (of $\phi(d)$ in the abelian case) to $\rho_E$. (See Stiller [5] for examples.)

One approach to the multiplicity problem is suggested by a similar looking multiplicity problem that goes back to

C. Chevalley and A. Weil, "Über das Verhalten der Integrale ersten Gattung bei Automorphismen des Funktionenkörpers," Abh. Math. Sem. Univ. Hamburg 10 (1934), 358-361.

A. Weil, "Über Matrizenringe auf Riemannschen Flächen und den Riemann-Rochschen Satz," Abh. Math. Sem. Univ. Hamburg 11 (1936) 110-115.

and in modern exposition:

J.F. Glazebrook and D.R. Grayson, "Galois representations on holomorphic differentials," preprint.

The set-up is

$\widetilde{X}/\mathbb{C}$ a curve of genus $\tilde{g}$

$G$ acts faithfully on $\widetilde{X}$ so $G \hookrightarrow \operatorname{Aut}(\widetilde{X})$

$G$ acts on $H^0(\widetilde{X}, (\Omega^1_{\widetilde{X}})^{\otimes q})$.

The results describe $H^0(\widetilde{X}, (\Omega^1_{\widetilde{X}})^{\otimes q})$ as a representation of $G$ for $q \geq 1$. Namely given an irreducible complex representation $V$ of $G$, a formula is given, in terms of local ramification invariants, for the multiplicity of $V$ in $H^0(\widetilde{X}, (\Omega^1_{\widetilde{X}})^{\otimes q})$.

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